Games on quantum objects

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Outline

1. Introduction to Quantum Mechanics (superposition and density matrices)
2. Density-matrix form of classical probability theory
3. Classical Game Theory (in density matrices)
4. Quantum game: the definition
5. Quantum game: the difference from the classical game
6. Possible future projects
7. References and acknowlodgement

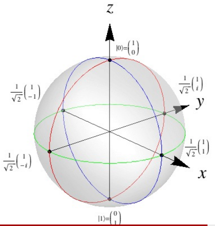
Quantum and classical probability theory

1. Quantum objects: Hilbert space $\mathcal{H}$, density matrix $(\rho \in \mathcal{N}(\mathcal{H}))$, observables $(H \in \mathcal{O}(\mathcal{H}))$, evolution operators $(U \in \mathcal{U}(\mathcal{H}))$
2. Evolution of states and average of observables:

$$\rho_f = U \rho_0 U^\dagger, \tag{1}$$

$$E = tr(\rho H). \tag{2}$$

3. Superposition principle: $\forall \left|\phi\right\rangle, \left|\psi\right\rangle \in \mathcal{H}, \alpha \left|\phi\right\rangle + \beta \left|\psi\right\rangle \in \mathcal{H}$. **Classical objects do not have such states**. For example, for a spin in *x*-direction state *up*:



$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \rho^q = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{3}$$

Quantum and classical probability theory, continued

1. Classical states can also be represented by, however, diagonal density matrices. For example, a coin with state *head* (another one with equal probability *head* or *tail*) can be written as

$$\rho_1^c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \rho_2^c = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4}$$

2. Its evolution and observable average satisfy respectively (1) and (2).
3. For instance, if we use the following rule to assign payoff: one gains one dollar for the *head* state and loses one otherwise, i.e. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then according to (2), we can get the average payoff

$$\langle A \rangle = tr(\rho_2^c A) = \frac{1}{2} tr(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = 0, \tag{5}$$

which reproduces exactly $\langle A \rangle = \sum_j A_j p_j$.

Classical Game

1. Game: conflict of interest among multiple players, to predict players' behavior, or mechanism design
2. Solution of games: pure and mixed strategies, **Nash equilibrium exists for all games only at the level of mixed strategies**
3. An example: Coin flipping, traditional abstract definition: set of strategies($S^{1,2} = \{I, X\}$), Payoff matrices($\{G^{1,2}\}$), a game is $\Gamma^c = \left(S^1 \otimes S^2, G^{1,2}\right)$,

$$G^1 = \left[\begin{array}{c|cc} & I & X \\ \hline I & 1 & -1 \\ X & -1 & 1 \end{array}\right] = -G^2, E^i = \left(p^1\right)^T G^i p^2, \tag{6}$$

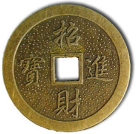
where $p^i = \left[p^i_I, p^i_X\right]^T$ is mixed strategy of the player i, **a vector to denote a probability distribution**.

Operational definition of quantum games

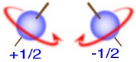
1. operational definition of a classical game: initial state $(\rho_0^c \in \mathcal{H})$, operators $(S^{1,2} = \mathcal{U}(\mathcal{H}), S = S^1 \otimes S^2)$, action of operators $(\mathcal{L}| : S \longrightarrow \mathcal{U}(\mathcal{H}))$, rules of payoff $(\{P^{1,2}\})$, a game is $\gamma^c = (\rho_0^c, \mathcal{U}(\mathcal{H}) \otimes \mathcal{U}(\mathcal{H}), \mathcal{L}, P^{1,2})$,

$$P^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -P^2, E^i = tr\left(\mathcal{L}\rho_0^c\mathcal{L}^\dagger P^i\right). \tag{7}$$

2. coins are replaced by spins as objects of games



$\longrightarrow$



3. Operational definition of a quantum game

$$\gamma^q = (\rho_0^q, \mathcal{U}(\mathcal{H}) \otimes \mathcal{U}(\mathcal{H}), \mathcal{L}, P^{1,2}), \tag{8}$$

Differs from classical game only at $\rho_0^c \to \rho_0^q$.

Difference between classical and quantum games?

1. First difference: much bigger set of strategies[1], $Z_2 \longrightarrow SU(2)$
2. However, classical games can also have countable or even uncountable many pure strategies
3. Mixed strategies are essential for solutions of games
4. What are mixed strategies in quantum game theory: classical probability distributions over the enlarged strategy set? Criticism from the classical game community[2].

A new abstract definition of classical and quantum games

1. Forget about objects, only strategies, **especially mixed strategies**
2. Classical games, $\Gamma^{c,new} = \left\{S^1 \otimes S^2, H^{1,2}\right\}$, mixed strategies $\rho^c = \rho^{c,1} \otimes \rho^{c,2}$, payoff $E^{1,2} = tr\left(\rho^c H^{1,2}\right)$. The coin flipping game:

$$\rho^{c,i} = \begin{bmatrix} p^i & 0 \\ 0 & 1-p^i \end{bmatrix}, H^1 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} = -H^2 \qquad (9)$$

3. Abstract definition of quantum game

$$\Gamma^{q,new} = \left\{S^1 \otimes S^2, H^{1,2}\right\}, \qquad (10)$$

Mixed strategy $\rho^q = \rho^{q,1} \otimes \rho^{q,2}$, payoff $E^{1,2} = tr\left(\rho^q H^{1,2}\right)$.

The real difference between classical and quantum game

1. quantum mixed strategy: off-diagonal elements of strategy density matrices and payoff matrices; classical game: all matrices are diagonal
2. The reason: superposition principle of strategies. Classically $\alpha I + \beta X$ is not a strategy, but, quantum operation $\frac{I+iX}{\sqrt{2}} \in SU(2)$ is still a well-defined strategy (i.e. terms like $-i\,|I)\,(X|$)
3. Example, spin flipping game:

$$H^1 = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & -i & 0 & 0 & i & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & -1 & -i & 0 & -1 & 0 & 0 & 1 & -i & 0 & 0 & i & 0 & -1 & -i & 0 \\
0 & i & -1 & 0 & i & 0 & 0 & -i & -1 & 0 & 0 & 1 & 0 & i & -1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & -i & 0 & 0 & i & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & -1 & -i & 0 & -1 & 0 & 0 & 1 & -i & 0 & 0 & i & 0 & -1 & -i & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & -i & 0 & 0 & i & 1 & 0 & 1 & 0 & 0 & 1 \\
i & 0 & 0 & i & 0 & i & 1 & 0 & 0 & -1 & i & 0 & i & 0 & 0 & i \\
0 & 1 & i & 0 & 1 & 0 & 0 & -1 & i & 0 & 0 & -i & 0 & 1 & i & 0 \\
0 & i & -1 & 0 & i & 0 & 0 & -i & -1 & 0 & 0 & 1 & 0 & i & -1 & 0 \\
-i & 0 & 0 & -i & 0 & -i & -1 & 0 & 0 & 1 & -i & 0 & -i & 0 & 0 & -i \\
1 & 0 & 0 & 1 & 0 & 1 & -i & 0 & 0 & i & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & -i & 1 & 0 & -i & 0 & 0 & i & 1 & 0 & 0 & -1 & 0 & -i & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & -i & 0 & 0 & i & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & -1 & -i & 0 & -1 & 0 & 0 & 1 & -i & 0 & 0 & i & 0 & -1 & -i & 0 \\
0 & i & -1 & 0 & i & 0 & 0 & -i & -1 & 0 & 0 & 1 & 0 & i & -1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & -i & 0 & 0 & i & 1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}.$$

Few possible new directions

1. Entangled initial states, ρ_0^q
2. Entanglement between players, implemented via entangled quantum states, its problem
3. measurements as operations (strategies) $S^i \supset \mathcal{U}(\mathcal{H})$
4. Nash equilibrium of quantum strategies (density matrices)?
5. Evolutionary equilibrium of quantum strategies?

Final Take-Home Message — Quantum games : classical game :: quantum mechanics : classical mechanics

References

1. D.A. Meyer, *Quantum Strategies*, Phys. Rev. Lett. **82**(1999), 1052-1055.
2. S.J. van Enk and R. Pike, *Classical rules in quantum games*, Phys. Rev. A **66**(2002), 024306.
3. J. Wu, Hamiltotian Formalism of Game Theory, arXiv: quant-ph/0501088.
4. H. Guo, J. Zhang and G.J. Koehler, *A survey of quantum games*, Decision Support Systems, **46**(2008), 318-332.

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