Game Theory, from a physicist?

A New Representation of Classical and Quantum Game Theory

Jinshan Wu

Department of Physics, Simon Fraser University

Outline

- Starting point, one example to show the whole idea
- Some Background of Physics
  - Quantum Mechanics
  - Statistical Mechanics
- Classical Game Theory
- New Representation of Classical and Quantum Game
- Only equivalent description, something new?

Prisoner's Dilemma as an example

- $(-2,-2),(-5,0),(0,-5),(-4,-4)$, four pairs of number for pure-strategy combinations (Deny, Confess)

Prisoner's Dilemma as an example

- $(-2,-2)\,,(-5,0)\,,(0,-5)\,,(-4,-4)$, four pairs of number for pure-strategy combinations (Deny, Confess)
- Weighted average for mixture-strategy combination

$$E^1 = (-2)\,p_d^1 p_d^2 + (-5)\,p_d^1 p_c^2 + (0)\,p_c^1 p_d^2 + (-4)\,p_c^1 p_c^2$$

Prisoner's Dilemma as an example

- $(-2,-2)\,,(-5,0)\,,(0,-5)\,,(-4,-4)$, four pairs of number for pure-strategy combinations (Deny, Confess)

- Weighted average for mixture-strategy combination

$$E^1 = (-2)\,p_d^1 p_d^2 + (-5)\,p_d^1 p_c^2 + (0)\,p_c^1 p_d^2 + (-4)\,p_c^1 p_c^2$$

- Turns into a matrix form G^i(tensor form for N-player game)

$$E^1 = \left[p_d^1, p_c^1\right] \begin{bmatrix} -2 & -5 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} p_d^2 \\ p_c^2 \end{bmatrix}$$

Prisoner's Dilemma as an example, continued

- Turns into a density matrix form ρ^i and H^i($(1,1)$-tensor for any-player game)

$$E^1 = Tr\left(\begin{bmatrix} p_d^1 & 0 \\ 0 & p_c^1 \end{bmatrix} \otimes \begin{bmatrix} p_d^2 & 0 \\ 0 & p_c^2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}\right)$$

$$= Tr\left(\begin{bmatrix} p_d^1 p_d^2 & 0 & 0 & 0 \\ 0 & p_d^1 p_c^2 & 0 & 0 \\ 0 & 0 & p_c^1 p_d^2 & 0 \\ 0 & 0 & 0 & p_c^1 p_c^2 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}\right)$$

Quantum Mechanics

Quantum Mechanics

- **State in Hilbert space $|\phi\rangle \in \mathcal{H}$, and base vectors $|\mu\rangle$**

$$|\phi\rangle = \sum_{\mu} \phi_{\mu} |\mu\rangle \,. \tag{1}$$

Quantum Mechanics

- **State in Hilbert space $|\phi\rangle \in \mathcal{H}$, and base vectors $|\mu\rangle$**

$$|\phi\rangle = \sum_{\mu} \phi_\mu |\mu\rangle . \tag{1}$$

- **Operators and physical quantities $A|_{\mathcal{H}\rightarrow\mathcal{H}}$**

$$|\psi\rangle = A|\phi\rangle \text{ and } \langle A\rangle = \langle\phi| A |\phi\rangle . \tag{2}$$

Quantum Mechanics

- State in Hilbert space $|\phi\rangle \in \mathcal{H}$, and base vectors $|\mu\rangle$

$$|\phi\rangle = \sum_{\mu} \phi_{\mu} |\mu\rangle . \tag{1}$$

- Operators and physical quantities $A|_{\mathcal{H}\to\mathcal{H}}$

$$|\psi\rangle = A|\phi\rangle \text{ and } \langle A\rangle = \langle\phi| A |\phi\rangle . \tag{2}$$

- Density matrix form for state

$$\begin{array}{c} \rho = |\phi\rangle\langle\phi| = \sum_{\mu,\nu} \phi_{\nu}^{*}\phi_{\mu} |\mu\rangle\langle\nu| \\ \text{but more general} \\ \rho = \sum_{\mu,\nu} \rho_{\mu\nu} |\mu\rangle\langle\nu| \end{array} \tag{3}$$

Quantum Mechanics, continued

- Dirac notation for operators $\langle\psi| \in \mathcal{H}^* |_{\mathcal{H}\to\mathbb{C}}$, so $|\phi\rangle\langle\psi|$ is a mapping from $\mathcal{H}$ to $\mathcal{H}$.

Quantum Mechanics, continued

- Dirac notation for operators $\langle\psi| \in \mathcal{H}^* |_{\mathcal{H}\to\mathbb{C}}$, so $|\phi\rangle\langle\psi|$ is a mapping from $\mathcal{H}$ to $\mathcal{H}$.
- Then any operator on $\mathcal{H}$ can be expanded as

$$A = \sum_{\mu\nu} A_{\mu\nu} |\mu\rangle\langle\nu| . \tag{4}$$

In such sense, we denote both state space ρ and operator space A as $\mathcal{H}$

Quantum Mechanics, continued

- Dirac notation for operators $\langle\psi| \in \mathcal{H}^* |_{\mathcal{H}\to\mathbb{C}}$, so $|\phi\rangle\langle\psi|$ is a mapping from $\mathcal{H}$ to $\mathcal{H}$.
- Then any operator on $\mathcal{H}$ can be expanded as

$$A = \sum_{\mu\nu} A_{\mu\nu} |\mu\rangle \langle\nu| \,. \tag{4}$$

In such sense, we denote both state space ρ and operator space A as $\mathcal{H}$

- Reduced density matrix

$$\rho_R^i = Tr_{-i}\left(\rho^S\right), \text{(if indepedent)} \rho^S = \prod_i \rho_R^i \tag{5}$$

Quantum Mechanics, continued

- In one word, Quantum Mechanics is a system
  - State (ρ) and Liouville Equation (or Schrödinger Equation),

$$\rho\left(t\right)=\hat{U}\rho_0\hat{U}^{\dagger}, \tag{6}$$

    in which $\hat{U}=\exp(-iHt)$, H is the Hamiltonian, a hermitian operator.
  - Physical quantities (A) and their values

$$\left\langle A\right\rangle = Tr\left(A\rho\right). \tag{7}$$

Statistical Mechanics

Statistical Mechanics

- Dynamical problem VS Thermal Dynamical problem

$$\rho_i \xrightarrow{\text{H, the Hamiltonian}} \rho_f$$
$$\rho_0 \xrightarrow{\text{heat bath, fluctuation}} \rho_{eq}$$

Statistical Mechanics

- Dynamical problem VS Thermal Dynamical problem

$$\rho_i \xrightarrow{\text{H, the Hamiltonian}} \rho_f$$
$$\rho_0 \xrightarrow{\text{heat bath, fluctuation}} \rho_{eq}$$

- Equilibrium distribution, ensemble theory, Canonical Ensemble

$$\rho_{eq} = \frac{1}{Z} e^{-\beta H} \text{ and } Z = Tr\left(e^{-\beta H}\right) \qquad (8)$$

Statistical Mechanics, continued

- Master equation and other pseudo-dynamical equations (for classical system)

$$\frac{d}{dt}\rho(x,t) = \sum_y W(y \to x)\,\rho(y) - \sum_y W(x \to y)\,\rho(x)\,. \tag{9}$$

Statistical Mechanics, continued

- Master equation and other pseudo-dynamical equations (for classical system)

$$\frac{d}{dt}\rho\left(x,t\right)=\sum_{y}W\left(y\rightarrow x\right)\rho\left(y\right)-\sum_{y}W\left(x\rightarrow y\right)\rho\left(x\right). \tag{9}$$

- For a quantum system, in a set of eigenvectors of H, we have

$$\rho_{eq}=\frac{1}{Z}e^{-\beta H}=\sum_{\mu}\frac{1}{Z}e^{-\beta E(\mu)}\left|\mu\right\rangle\left\langle\mu\right|. \tag{10}$$

Classical Game Theory

- **Solution of a game**

Classical Game Theory

- **Solution of a game**
- **Nash Equilibrium and Nash Theorem**

$$E^i\left(\vec{P}^1_{eq},\cdots,\vec{P}^i_{eq},\cdots,\vec{P}^N_{eq}\right)\geq E^i\left(\vec{P}^1_{eq},\cdots,\vec{P}^i,\cdots,\vec{P}^N_{eq}\right) \tag{11}$$

Classical Game Theory

- Solution of a game
- Nash Equilibrium and Nash Theorem

$$E^i\left(\vec{P}^1_{eq},\cdots,\vec{P}^i_{eq},\cdots,\vec{P}^N_{eq}\right) \geq E^i\left(\vec{P}^1_{eq},\cdots,\vec{P}^i,\cdots,\vec{P}^N_{eq}\right) \tag{11}$$

- Evolutionary Game and its relation with static game

$$\left(\vec{P}^1_i,\cdots,\vec{P}^N_i\right) \xrightarrow{\underline{Evolution}} \left(\vec{P}^1_f,\cdots,\vec{P}^N_f\right) \Leftrightarrow \left(\vec{P}^1_{eq},\cdots,\vec{P}^N_{eq}\right) \tag{12}$$

Classical Game Theory, continued

- Cooperative Game and its relation with static game

$$N = \bigcup_{j=1}^{K} N^j, \rho^S = \prod_{j=1}^{K} \rho^j \neq \prod_{i=1}^{N} \rho^i \tag{13}$$

Classical Game Theory, continued

- Cooperative Game and its relation with static game

$$N = \bigcup_{j=1}^{K} N^j, \rho^S = \prod_{j=1}^{K} \rho^j \neq \prod_{i=1}^{N} \rho^i \qquad (13)$$

- Questions: Calculation of NE, Unstable NE as a solution, Refinement of NEs

Comparison between Game Theory and Physics

- Comparison with Quantum Mechanics
  - Hamiltonian
  - State, Density Matrix
  - Dynamical Equation

Comparison between Game Theory and Physics

- Comparison with Quantum Mechanics
  - Hamiltonian
  - State, Density Matrix
  - Dynamical Equation

- Comparison with Statistical Mechanics
  - Ensemble, Distribution, Thermal Equilibrium
  - Pseudo-dynamical Equation

The New Representation of Classical Game

- Traditional Classical Game

$$\Gamma^{c} = \left(\prod_{i=1}^{N} \otimes S_i, \left\{ G^i \right\} \right), \tag{14}$$

- The new form

$$\Gamma^{c,new} = \left(\prod_{i=1}^{N} \otimes S_i, \left\{ H^i \right\} \right). \tag{15}$$

The New Representation of Classical Game

- Traditional Classical Game

$$\Gamma^c = \left(\prod_{i=1}^{N} \otimes S_i, \left\{ G^i \right\} \right), \tag{14}$$

- The new form

$$\Gamma^{c,new} = \left(\prod_{i=1}^{N} \otimes S_i, \left\{ H^i \right\} \right). \tag{15}$$

- The relations, and question about non-zero off-diagonal elements of H^i

$$\begin{array}{ccc} \prod_i \vec{P}^i & \longrightarrow & \rho^S \\ G^i & \longrightarrow & H^i = \sum_{SS'} G^i_S \delta_{SS'} \left| S \right\rangle \left\langle S' \right| \end{array} \tag{16}$$

The New Representation, continued

- **Reduced Payoff Matrix**

$$\begin{aligned} H_R^i &= Tr_{-i}\left(\rho^1 \cdots \rho^{i-1}\rho^{i+1} \cdots \rho^N H^i\right) \\ E^i &= Tr^i\left(\rho^i H_R^i\right) \end{aligned} \tag{17}$$

The New Representation, continued

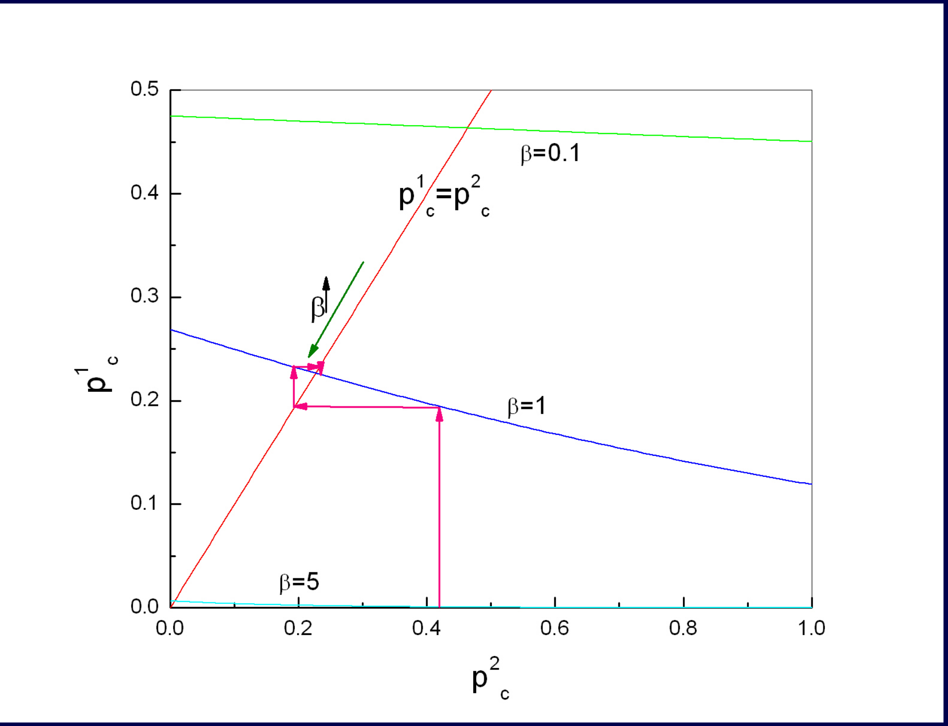
- Reduced Payoff Matrix

$$\begin{aligned} H^i_R &= Tr_{-i}\left(\rho^1 \cdots \rho^{i-1}\rho^{i+1}\cdots\rho^N H^i\right) \\ E^i &= Tr^i\left(\rho^i H^i_R\right) \end{aligned} \tag{17}$$

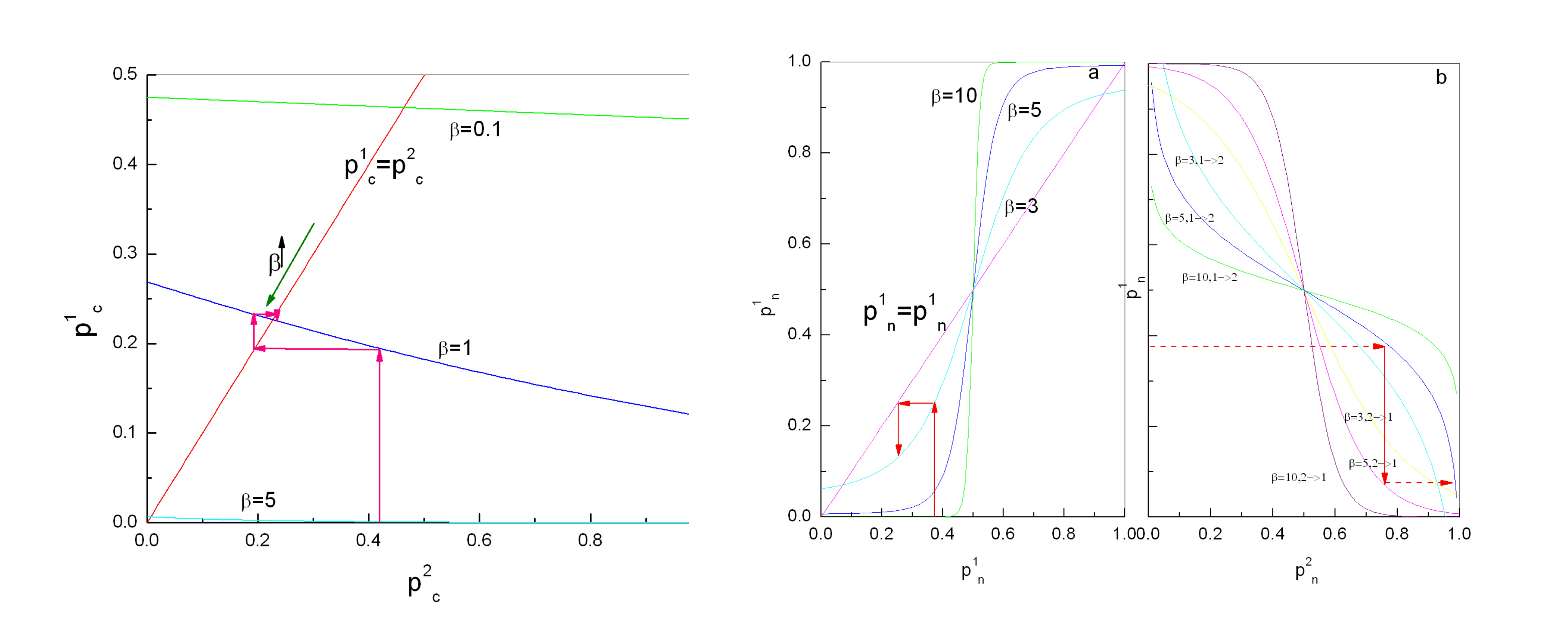
- Pseudo-dynamical Equation

$$\rho^i\,(t+1) = \frac{e^{\beta H^i_R(t)}}{Tr^i(e^{\beta H^i_R(t)})} \tag{18}$$

The New Representation, Application



The New Representation, Application



- Prisoner’s Dilemma
- Hawk-Dove Game

The New Representation of Quantum Game

- **Manipulative Definition**

$$\Gamma^{q,o} = \left(\rho_o^q \in \mathbb{H}^q, \prod_{i=1}^{N} \otimes \mathbb{H}^i, \mathcal{L}, \left\{P^i\right\}\right)$$

$$E^i = Tr\left(P^i \mathcal{L}\left(\cdots, \hat{U}^i, \cdots\right) \rho_0^q \mathcal{L}^\dagger\left(\cdots, \hat{U}^i, \cdots\right)\right) \tag{19}$$

The New Representation of Quantum Game

- Manipulative Definition

$$\Gamma^{q,o} = \left(\rho_o^q \in \mathbb{H}^q, \prod_{i=1}^{N} \otimes \mathbb{H}^i, \mathcal{L}, \left\{P^i\right\}\right)$$

$$E^i = Tr\left(P^i \mathcal{L}\left(\cdots, \hat{U}^i, \cdots\right) \rho_0^q \mathcal{L}^\dagger \left(\cdots, \hat{U}^i, \cdots\right)\right) \quad (19)$$

- Natural base strategies of operator space

The New Representation of Quantum Game

- Manipulative Definition

$$\Gamma^{q,o} = \left(\rho_o^q \in \mathbb{H}^q, \prod_{i=1}^N \otimes \mathbb{H}^i, \mathcal{L}, \left\{P^i\right\}\right)$$
$$E^i = Tr\left(P^i \mathcal{L}\left(\cdots, \hat{U}^i, \cdots\right) \rho_0^q \mathcal{L}^\dagger \left(\cdots, \hat{U}^i, \cdots\right)\right) \tag{19}$$

- Natural base strategies of operator space
- Abstract Form

$$\Gamma^q = \left(\prod_{i=1}^N \otimes S_i^q, \left\{H^i\right\}\right)$$
$$E^i = Tr\left(\rho^S H^i\right) \tag{20}$$

The New Representation, continued

- Quantum Nash Equilibrium
  - General NE in system state space

$$E^i\left(\rho_{eq}^S\right) \geq E^i\left(Tr^i\left(\rho_{eq}^S\right)\cdot\rho^i\right), \forall\rho^i, \forall i. \quad (21)$$

  - NE in direct-product strategy space

$$E^i\left(\rho_{eq}^1, \cdots, \rho_{eq}^i, \cdots, \rho_{eq}^N\right) \geq E^i\left(\rho_{eq}^1, \cdots, \rho^i, \cdots, \rho_{eq}^N\right) \quad (22)$$

  - Strong dominant state

$$E^i\left(\rho_{eq}^S\right) \geq E^i\left(\rho^S\right), \forall\rho^S. \quad (23)$$

The New Representation, continued

- Quantum Nash Equilibrium
  - General NE in system state space

$$E^i\left(\rho_{eq}^S\right) \geq E^i\left(Tr^i\left(\rho_{eq}^S\right)\cdot\rho^i\right), \forall\rho^i, \forall i. \qquad (21)$$

  - NE in direct-product strategy space

$$E^i\left(\rho_{eq}^1, \cdots, \rho_{eq}^i, \cdots, \rho_{eq}^N\right) \geq E^i\left(\rho_{eq}^1, \cdots, \rho^i, \cdots, \rho_{eq}^N\right) \qquad (22)$$

  - Strong dominant state

$$E^i\left(\rho_{eq}^S\right) \geq E^i\left(\rho^S\right), \forall\rho^S. \qquad (23)$$

- Non-direct-product NE means Cooperative?

Quantum Game, Application I — QPF Game

- Manipulative Definition
  - Base vectors of state space: $|1\rangle = (1,0)^T, |-1\rangle = (0,1)^T$
  - Initial state of the quantum object: $\rho_0^q = |1\rangle\langle 1|$
  - Strategies of quantum players: $I, \sigma_x, \sigma_y, \sigma_z$ and their combination
  - Payoff scale: player 1 win 1 when $|1\rangle$ lose 1 when $|-1\rangle$, inverse for player 2.
  - So

$$E^1 = Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U^2 U^1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(U^2 U^1\right)^\dagger\right) = -E^2$$

Quantum Game, Application I — continued

- Payoff Matrix with non-zero off-diagonal elements, for example, $\left\langle I, I \left| H^1 \right| \sigma_x, I \right\rangle = 1$, while classically no such situation that looking on the left side, both players choose I, but according to right side, not.

Quantum Game, Application I — continued

- **Payoff Matrix with non-zero off-diagonal elements, for example, $\langle I, I \left| H^1 \right| \sigma_x, I \rangle = 1$, while classically no such situation that looking on the left side, both players choose I, but according to right side, not.**

$$\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0 & i & 1 & 0 & 1 & 0 & 0 & -i & 0 & 1 \\
0 & -1 & 0 & i & -1 & 0 & 1 & 0 & 0 & -1 & 0 & i & i & 0 & -i & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & i & 1 & 0 & 1 & 0 & 0 & -i & 0 & 1 \\
0 & -i & 0 & -1 & -i & 0 & i & 0 & 0 & -i & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & i & -1 & 0 & 1 & 0 & 0 & -1 & 0 & i & i & 0 & -i & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & i & 1 & 0 & 1 & 0 & 0 & -i & 0 & 1 \\
0 & 1 & 0 & -i & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -i & -i & 0 & i & 0 \\
-i & 0 & -i & 0 & 0 & -i & 0 & 1 & -i & 0 & -i & 0 & 0 & -1 & 0 & -i \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & i & 1 & 0 & 1 & 0 & 0 & -i & 0 & 1 \\
0 & -1 & 0 & i & -1 & 0 & 1 & 0 & 0 & -1 & 0 & i & i & 0 & -i & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & i & 1 & 0 & 1 & 0 & 0 & -i & 0 & 1 \\
0 & -i & 0 & -1 & -i & 0 & i & 0 & 0 & -i & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & -i & 0 & -1 & -i & 0 & i & 0 & 0 & -i & 0 & -1 & -1 & 0 & 1 & 0 \\
i & 0 & i & 0 & 0 & i & 0 & -1 & i & 0 & i & 0 & 0 & 1 & 0 & i \\
0 & i & 0 & 1 & i & 0 & -i & 0 & 0 & i & 0 & 1 & 1 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & i & 1 & 0 & 1 & 0 & 0 & -i & 0 & 1
\end{bmatrix}$$

Application II — Quantum Battle of Sexes

- **An Artificial Game and its classical limit ($\epsilon_1 > \epsilon_2$)**

$$H^1 = \begin{bmatrix} \epsilon_1 & 0 & 0 & \epsilon_1 \\ 0 & \epsilon_2 & \epsilon_2 & 0 \\ 0 & \epsilon_2 & \epsilon_2 & 0 \\ \epsilon_1 & 0 & 0 & \epsilon_1 \end{bmatrix} = H^2,$$

Application II — Quantum Battle of Sexes

- **An Artificial Game and its classical limit ($\epsilon_1 > \epsilon_2$)**

$$H^1 = \begin{bmatrix} \epsilon_1 & 0 & 0 & \epsilon_1 \\ 0 & \epsilon_2 & \epsilon_2 & 0 \\ 0 & \epsilon_2 & \epsilon_2 & 0 \\ \epsilon_1 & 0 & 0 & \epsilon_1 \end{bmatrix} = H^2,$$

$$H^{1,c} = \begin{bmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{bmatrix} = H^{2,c}$$

Application II — Quantum Battle of Sexes

- **An Artificial Game and its classical limit ($\epsilon_1 > \epsilon_2$)**

$$H^1 = \begin{bmatrix} \epsilon_1 & 0 & 0 & \epsilon_1 \\ 0 & \epsilon_2 & \epsilon_2 & 0 \\ 0 & \epsilon_2 & \epsilon_2 & 0 \\ \epsilon_1 & 0 & 0 & \epsilon_1 \end{bmatrix} = H^2,$$

$$H^{1,c} = \begin{bmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{bmatrix} = H^{2,c}$$

- **Strong Dominant State, an entangled strategy state**

$$\rho_M^S = (|BB\rangle + |SS\rangle)(\langle BB| + \langle SS|) \neq \rho^1 \otimes \rho^2$$

Conclusions and Discussions: Something new?

Conclusions and Discussions: Something new?

- At least, an equivalent description of classical game
- At least, a general form of quantum game, a truly quantum game, not in the scope of traditional G^i

Conclusions and Discussions: Something new?

- At least, an equivalent description of classical game
- At least, a general form of quantum game, a truly quantum game, not in the scope of traditional G^i
- Unified framework for both classical and quantum game, so easier to transfer ideas between them

Conclusions and Discussions: Something new?

- At least, an equivalent description of classical game
- At least, a general form of quantum game, a truly quantum game, not in the scope of traditional G^i
- Unified framework for both classical and quantum game, so easier to transfer ideas between them
- Possible way leading to Evolutionary Game and Cooperative Game

Any Future?

Any Future?

- Quantum Nash Equilibrium Proposition, but, without a general proof (hopefully in near future)

Any Future?

- Quantum Nash Equilibrium Proposition, but, without a general proof (hopefully in near future)
- Better Dynamical Equations for classical game and one applicable equation for quantum game (still open problem in Physics)

Any Future?

- Quantum Nash Equilibrium Proposition, but, without a general proof (hopefully in near future)
- Better Dynamical Equations for classical game and one applicable equation for quantum game (still open problem in Physics)
- Application onto more specific games (which one?)

Any Future?

- Quantum Nash Equilibrium Proposition, but, without a general proof (hopefully in near future)
- Better Dynamical Equations for classical game and one applicable equation for quantum game (still open problem in Physics)
- Application onto more specific games (which one?)
- Application onto some tough theoretical problems (where?)

Reference and Thank You All, Question Time Now

Reference and Thank You All, Question Time Now

- Jinshan Wu, arXiv:quant-ph/0404159.
- S. J. van Enk and R. Pike, Phys. Rev. A 66(2002), 024306.
- D.A. Meyer, Phys. Rev. Lett. 82(1999), 1052.
- J. Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett, 83(1999), 3077.
- Jinshan Wu, arXiv:quant-ph/0405003.
- Jinshan Wu, arXiv:quant-ph/0405032.