A new mathematical representation of Game Theory II

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Abstract

In another paper with the same name[1], we proposed a new representation of Game Theory, but most results are given by specific examples and argument. In this paper, we try to prove the conclusions as far as we can, including a proof of equivalence between the new representation and the traditional Game Theory, and a proof of Classical Nash Theorem in the new representation. And it also gives manipulation definition of quantum game and a proof of the equivalence between this definition and the general abstract representation. A Quantum Nash Proposition is proposed but without a general proof. Then, some comparison between Nash Equilibrium (NE) and the pseudo-dynamical equilibrium (PDE) is discussed. At last, we investigate the possibility that whether such representation leads to truly Quantum Game, and whether such a new representation is helpful to Classical Game, as an answer to the questions in [2]. Some discussion on continuous-strategy games are also included.

Key Words: Game Theory, Nash Equilibrium, Quantum Game, Continuous Strategy.

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1 Introduction

In [1], we have pointed out that all ideas and concepts in traditional framework of Game Theory can be translated into our new representation, by some specific examples of discrete-strategy games. Now by giving some proof, especially a proof of Nash Theorem, we wish to confirm that this new representation can express every idea in traditional framework of classical game.

On the other hand, for quantum games, as pointed out in [2], two questions should be answered when a quantum framework or a quantized version of classical game is discussed to compare with the corresponding classical game. The first one is whether the new approach is helpful to solve the classical one, the real original classical game, not the quantized version. The second one is whether the quantized version is a truly quantum mechanics problem with independent meaning other than the corresponding classical one. In this paper, we try to give a positive answer to those two questions, although frankly speaking, our new representation of game theory[1] is not exactly the same with quantum games in [3, 4].

In the next section (§2), we shortly review the structure of our new representation and results when applied onto discrete-strategy game. Then classical Nash Theorem in the new representation is proved in section §4 and compared with our Pseudo-dynamical Equilibrium (PDE)(§5). In §6, Quantum Game is defined from the manipulation level starting from quantum object and quantum operators. Also NE and Nash Theorem for quantum game is proposed there, but not proved. The last part (§7) is devoted to answer the two questions mentioned above, while a short summary of the results in this paper is also included.

2 The new representation of classical game

Traditionally, a classical static non-cooperative game is defined as

$$\Gamma^c = \left(\prod_{i=1}^{N} \otimes S_i, \left\{G^i\right\}\right), \tag{1}$$

in which S_i is the L_i-element strategy set of player i, and G^i is a mapping from $\prod_{i=1}^{N} \otimes S_i$ to $\mathbb{R}$. A state of player i can be a mixture strategy as

$$\vec{P}^i = \left(p^1_1, \cdots, p^1_\mu, \cdots, p^1_{L_i}\right)^T, \tag{2}$$

in which p^i_μ is the probability that player i choose strategy μ from the set S_i. The payoff value of player i is

$$E^i\left(\vec{P}^1, \cdots, \vec{P}^N\right) = \sum_{s^1_\mu, \cdots, s^N_\nu} G^i_{s^1_\mu \cdots s^N_\nu} p^1_\mu \cdots p^N_\nu. \tag{3}$$

So for state vector $\vec{P}^i$, $\left\{G^i\right\}$ is a set of $(0, N)$-tensor. A Nash Equilibrium state $\vec{P}^1_{eq}, \cdots, \vec{P}^N_{eq}$ is defined that

$$E^i\left(\vec{P}^1_{eq}, \cdots, \vec{P}^i_{eq}, \cdots, \vec{P}^N_{eq}\right) \geq E^i\left(\vec{P}^1_{eq}, \cdots, \vec{P}^i, \cdots, \vec{P}^N_{eq}\right), \forall i, \forall \vec{P}^i. \tag{4}$$

For a continuous-strategy game, $\vec{P}^i$, the state vector of player i, will be a probability distribution function on the continuous strategy set, and all summations turn into integral.

In our new representation, we defined base vector and inner product in strategy sets so as to form them as Hilbert space, and then the system state space is the direct product space of all single-player spaces. The base vectors of player i's strategy space are chosen as all the pure strategies, and denoted as $\left|s_{\mu}^{i}\right\rangle$, the inner product is defined as

$$\left\langle s_{\mu}^{i} \middle| s_{\nu}^{i}\right\rangle = \delta\mu\nu. \tag{5}$$

So base vectors of system state space are

$$|S\rangle = \left|s_{\mu}^{1}\right\rangle \cdots \left|s_{\nu}^{N}\right\rangle \triangleq \left|s_{\mu}^{1}, \cdots, s_{\nu}^{N}\right\rangle. \tag{6}$$

Then, a state of player i expressed in density matrix form is

$$\rho^{i} = \sum_{\mu} p_{\mu}^{i} |\mu\rangle \langle\mu|. \tag{7}$$

So a system state is

$$\rho^{S} \triangleq \prod_{i}^{N} \rho^{i} = \sum_{S} p_{\mu}^{1} \cdots p_{\nu}^{N} |S\rangle \langle S|. \tag{8}$$

According to Quantum Mechanics, the expectation value should be calculated by $E = Tr\left(\rho^{S} H\right)$, in which H is a quantum operator defined on system state space. We want to keep the same mathematical form, so the payoff value of player i must be

$$E^{i} = Tr\left(\rho^{S} H^{i}\right). \tag{9}$$

Now we need to define a payoff matrix H^{i} to guarantee that the payoff from equ(9) are equivalent with the payoff from equ(3). Under such requirement, we find

$$H_{SS'}^{i} = G_{S}^{i} \delta_{SS'} \quad \text{or} \quad H^{i} = \sum_{SS'} G_{S}^{i} \delta_{SS'} \left| S \right\rangle \left\langle S' \right|, \tag{10}$$

in which

$$G_{S}^{i} = G_{s_{\mu}^{1} \cdots s_{\nu}^{N}}^{i}. \tag{11}$$

It's easy to check the equivalence,

$$\begin{array}{rcl} E^{i} = Tr\left(\rho^{S} H^{i}\right) & = & \sum_{S'} \left\langle S' \middle| \rho^{S} H^{i} \middle| S' \right\rangle \\ & = & \sum_{S',S} \left\langle S' \middle| \rho^{S} \middle| S \right\rangle \left\langle S \middle| H^{i} \middle| S' \right\rangle \\ & = & \sum_{S} \left\langle S \middle| \rho^{S} \middle| S \right\rangle H_{SS}^{i} \\ & = & \sum_{S} \rho_{SS}^{S} H_{SS}^{i} \\ & = & \sum_{S} p_{\mu}^{1} \cdots p_{\nu}^{N} G_{s_{\mu}^{1} \cdots s_{\nu}^{N}}^{i} \end{array}. \tag{12}$$

Now in our new representation, a game is redefined as

$$\Gamma^{c,new} = \left(\prod_{i=1}^{N} \otimes S_{i}, \left\{H^{i}\right\}\right). \tag{13}$$

So we have

Theorem I Classical game $\Gamma^{c,new}$ is equivalent with game Γ^{c}.

NE is redefined as ρ_{eq}^{S} that

$$E^i\left(\rho_{eq}^1\cdots\rho_{eq}^i\cdots\rho_{eq}^N\right)\geq E^i\left(\rho_{eq}^1\cdots\rho^i\cdots\rho_{eq}^N\right),\forall i,\forall\rho^i. \tag{14}$$

A reduced payoff matrix, which means the payoff matrix when all other players' states are fixed, is defined as

$$H_R^i=Tr_{-i}\left(\rho^1\cdots\rho^{i-1}\rho^{i+1}\cdots\rho^N H^i\right), \tag{15}$$

in which $Tr_{-i}\left(\cdot\right)$ means to do the trace in the space except the one of player i. If a trace in player i's space is needed, we denote it as $Tr^i\left(\cdot\right)$. Payoff value can also be calculated by the reduced matrix as

$$E^i=Tr^i\left(\rho^i H_R^i\right). \tag{16}$$

3 Continuous strategy space

For a continuous-strategy game, we need to replace summation $\left(\sum_\mu\right)$ with integral $\left(\int d\mu\right)$ and to replace probability p_μ^i with probability density function $p^i\left(\mu\right)$. And then, the inner product will be

$$\left\langle\mu^i\left|\nu^i\right.\right\rangle=\delta\left(\mu^i-\nu^i\right). \tag{17}$$

In fact, before we define this, we have to claim what's $d\mu$, the measure of μ. But here, let's say our continuous strategies are something like price, so that a nature measure is predefined. Another problem is the normalization condition. As Quantum Mechanics, we have to deal with the term like $\left\langle\mu\left|\mu\right.\right\rangle$. In Quantum Mechanics, it doesn't matter if wave function is used as a description of state. On the other hand, from equ(12), we know that because H^i has only diagonal terms, only diagonal part of ρ^i will effect the payoff. This means that if another density matrix but with the same diagonal part is used to describe state, it will give the same payoff value. In Quantum Mechanics viewpoint, a pure state described by a wave function can be used to replace the density matrix of mixture state if

$$\rho_{ss}^i=\phi^*(s)\phi(s). \tag{18}$$

Then, in a continuous-strategy game, a state of player i is $\left|\phi\right\rangle=\int d\mu\phi(\mu)\left|\mu\right\rangle$, or density matrix form,

$$\rho^i=\int\int d\mu d\nu\phi^*(\nu)\phi(\mu)\left|\mu\right\rangle\left\langle\nu\right|. \tag{19}$$

But only diagonal term $\phi^*(\mu)\phi(\mu)$ will effect the payoff. The normalization condition $\int d\mu\left\langle\mu\left|\rho^i\right|\mu\right\rangle=1$ gives

$$\int d\mu\phi^*(\mu)\phi(\mu)=1. \tag{20}$$

The traditional payoff function and the relation with payoff value is

$$E^i=\int dSp^1\left(s^1\right)\cdots p^N\left(s^N\right)G^i\left(s^1,\cdots,s^N\right), \tag{21}$$

in which, according to eu(18), $p^i(\mu) = \left|\phi^i(\mu)\right|^2$. And in our new representation,

$$E^i = \int dS \left\langle S\right| \rho^1 \cdots \rho^N H^i \left|S\right\rangle, \tag{22}$$

in which

$$H^i = \int\int dS dS^{'} \left|S\right\rangle\left\langle S^{'}\right| G^i(S)\,\delta(S - S^{'}). \tag{23}$$

It's easy to prove equ(21) and equ(22) give the same payoff value. Although here all formulas are derived in continuous strategy space form, for discrete strategy games, pure state with condition $\left|\phi^i_\mu\right|^2 = p^i_\mu$ can also be used to replace density matrix form of mixture state and give the same payoff.

4 Proof of Nash Theorem

Nash Theorem proves the existence of NE. For a game defined by equ(1), equilibrium states defined by equ(4) always exist. Now in our new representation, Nash Theorem is reexpressed as

Theorem II For a game defined by equ(13), equilibrium states defined by equ(14) always exist.

Proof Just following the idea of Nash's proof, first, we define a mapping, and prove the existence of the fixed points of this mapping. Then we will prove the fixed points are NE.

A mapping is defined as

$$\left(\rho^{1,'}, \cdots, \rho^{N,'}\right) = \mathcal{T}\left(\rho^1, \cdots, \rho^N\right), \tag{24}$$

in which,

$$\rho^{i,'} = \frac{\rho^i}{1 + Tr\left(\Delta E^i\right)} + \frac{\Delta E^i}{1 + Tr\left(\Delta E^i\right)}, \tag{25}$$

in which

$$\Delta E^i = Max\left\{0, H^i_R - E^i\left(\rho^S\right) I^i\right\}, \tag{26}$$

in which Max means to get the bigger one between every element. First, as a physicist usually does, let's show the fixed points are NE. Denote the fixed points as $\rho^S_{eq} = \prod_{i=1}^N \rho^i_{eq}$, then

$$\rho^i_{eq} Tr\left(\Delta E^i\right) = \Delta E^i. \tag{27}$$

It's easy to know that

$$\Delta E^i = 0 \tag{28}$$

is one of the solutions. Let's suppose $\Delta E^i \neq 0$, because every element is bigger than 0, $Tr\left(\Delta E^i\right) > 0$. Therefor, from equ(27), if the diagonal element of $\rho^i_{eq,\mu\mu} > 0$, $\Delta E^i_{\mu\mu} > 0$. Then from equ(26), the definition of ΔE^i, because it's bigger than zero,

$$H^i_{R,\mu\mu} > E^i\left(\rho^S\right) = \sum_\nu \rho^i_{eq,\nu\nu} H^i_{R,\nu\nu}, \forall \rho^i_{eq,\mu\mu} > 0.$$

So the weighted average of $H^i_{R,\mu\mu}$,

$$\sum_\mu \rho^i_{eq,\mu\mu} H^i_{R,\mu\mu} > \sum_\nu \rho^i_{eq,\nu\nu} H^i_{R,\nu\nu}$$

That's impossible. So $\Delta E^i = 0$ is the only solution. Therefor, from equ(26),

$$E^i\left(\rho^S_{eq}\right) \geq H^i_{R,\mu\mu}, \forall \mu.$$

And then, since it's bigger than every element of H^i_R, it's bigger than any weighted average of them, so it's NE. All players can't get more payoff by adjusting their own states independently. Now we claim that this is a continuous and onto mapping from system Hilbert space to itself. So it has fixed points. This detailed proof is neglected here.

Now we have shown that the existence of Nash Equilibrium in our new representation. The mapping defined here can be regarded as an iteration starting from any arbitrary initial system state. But does it converge onto the fixed points? Are the fixed points stable? From NE and the proof of Nash Theorem, no thing we can say about this question. The real experience in application of Game Theory shows that sometime the NE is not stable. If they are unstable, they are not very meaningful to be regarded as a prediction of the game.

However, our representation here is quite similar with Quantum Mechanics and Statistical Mechanics. We have state vector or density matrix, and we have dynamical variables such as payoff matrix and reduced payoff matrix, which look very like a Hamiltonian. The only thing missing here is a dynamical equation, which determines the evolution of state.

5 Comparison between NE and PDE

In order to give an evolutionary equation, we recall Master Equation for probability distribution function,

$$\frac{dp^i(x,t)}{dt} = \sum_{x'} w\left(x' \rightarrow x\right) p^i\left(x',t\right) - \sum_{x'} w\left(x \rightarrow x'\right) p^i(x,t), \tag{29}$$

in which, we suppose the transition rate is

$$w\left(x' \rightarrow x\right) = \frac{e^{\beta\left[E^i(x) - E^i\left(x'\right)\right]}}{\sum_y e^{\beta\left[E^i(y) - E^i\left(x'\right)\right]}} = \frac{e^{\beta E^i(x)}}{\sum_y e^{\beta E^i(y)}}.$$

The Master Equation here is actually N related equations, because E^i depends on H^i_R, which is determined by other players' state. From background of Statistical Mechanics, we know, if it's a single equation, or we say, E^i is independent on other players' state, the equilibrium state when $\frac{dp^i(x,t)}{dt} = 0$ will be

$$p^i(x,\infty) = \frac{e^{\beta E^i(x)}}{\sum_{x'} e^{\beta E^i(x')}}.$$

But, unfortunately, here all equations are related. So we make another assumption that the time scale of a single Master Equation is much smaller than the time scale of the related equations. In physics, this means that we let the single equation evolute to equilibrium first, then we feedback the equilibrium state into all other equations and so on. Under such assumption, we will get another N related equations from equ(29),

$$p^i(x,t+1) = \frac{e^{\beta E^i(x,t)}}{\sum_{x'} e^{\beta E^i(x',t)}}.$$

In density matrix notation,

$$\rho^i(t+1) = \frac{e^{\beta H_R^i(t)}}{Tr^i(e^{\beta H_R^i(t)})}, \tag{30}$$

very similar with the Boltzman distribution for canonical ensemble in Statistical Mechanics. Now we have an evolutionary equation although we don't know it can give some information about the game solution or not. It's not on the basis of first principle, however, as in Statistical Mechanics, since we wish it will give reasonable game solution as equilibrium state, we name it pseudo-dynamical equation and name the equilibrium state if possible Pseudo-Dynamical Equilibrium (PDE).

In such evolution, first, we choose an initial state for every player, at every step, start from player i, calculate H_R^i, get the new state of player i by equ(30), then feed it into other players' reduced payoff matrix to get their new states. Repeat such step till some fixed pattern if it's possible. The existence of such pattern is not proved, and the specific order of choosing which player first may affect such pattern. So the whole thing is still open, and should be investigated further. Even the equation itself is derived by two assumptions, first the Master Equation (29) and second the time scale assumption. The intuitive meaning of such iteration equation is that every player decides its own response to all other players according to the possible payoff, but instead of choosing the best one uses a distribution function, then other players repeat such iteration and so on.

The privilege of our representation is that because we use payoff matrix (a $(1,1)$-tensor) and reduced payoff matrix to calculate the payoff value, when they are putted onto exponential function, they will give a naturally defined density matrix of players' states. Compared with the mapping of equ(25), iteration process equ(30) can be approximately reexpressed as

$$\rho^i(t+1) = \frac{\rho^i(t)}{1+Tr^i(e^{\beta H_R^i(t)})} + \frac{e^{\beta H_R^i(t)}}{1+Tr^i(e^{\beta H_R^i(t)})}, \tag{31}$$

because usually $Tr^i(e^{\beta H^i(t)}) \gg 1$. So the difference between the mapping equ(25) and our iteration is that matrix $e^{\beta H^i_R(t)}$ is used to replace matrix ΔE^i. We wish such replacement will not change the idea of NE so far that it still can give information of game solution. On the other hand, besides the similarity with the mapping, the iteration process here looks very reasonable and comparable with the real game process. Everyone chooses initial state first, then decides the best response according to states of other players, and then iterates such process.

However, whatever it looks like, the only test is whether it will give reasonable game solution or not. A theoretical comparison between PDE and NE is ongoing in our work plan, but here, as far as the well-known specific games we tried, it gives quite good results[1]. And in some cases, when unstable NE exists, our iteration gives some pattern such as a jumping between some NE states, in other cases, when stable NEs exist, it end at one point of the NEs depending on initial state. It seems that even the iteration process itself is meaningful. In such cases, stable NEs can be regarded as the end results of the iteration process. It's quite amazing, but still waiting for more exploration.

6 Quantum Game and Quantum Nash Equilibrium

From the proof of equivalence, equ(12), we know that because payoff matrix H^i has only diagonal term, only diagonal term of ρ^S comes into the expression of payoff value. And similarly, if ρ^S has only the diagonal term, only diagonal term of H^i will be effective. This property implies two things. The first, as we did for continuous strategy games, a wave function can be used to replace distribution function. Second, if in some games, a payoff function with non-zero off-diagonal elements **and** a non-zero off-diagonal density matrix, are required, they will be totally **new games**. Effect of the first aspect will be discussed in the last section (§7). In this section, we try to find some manipulative examples of the new games.

On the other hand, let's suppose players can make use of quantum operator as strategy. We have a quantum object, every player applies a quantum operator on it and then the payoff of every player is determined by the end state of quantum object. In fact, this game use the same idea of classical game, but with the quantum object and quantum strategies. As in [1], using Quantum Penny Flip Game[3] as example, the quantum object is a spin, the quantum strategies are all unitary matrix acting on the spin, and the payoff is determined by the end state of the spin. Compared with classical games, in quantum game, base strategies still can be defined, even quite natural. State space of the quantum object has Q base vectors $\{|\mu\rangle\}$, so the quantum operator has the form

$$\hat{U} = \sum_{\mu\nu} U_{\mu\nu} |\mu\rangle \langle\nu| . \tag{32}$$

Therefor, $\{|\mu\rangle \langle\nu|\}$ can be regarded as $Q \times Q$ base vectors of the operator space. And because the operator space is still a Hilbert space, we even can define the inner product as

$$\left(\hat{U}, \hat{V}\right) = Tr\left(\hat{U}^{\dagger}\hat{V}\right) \tag{33}$$

so that $\{|\Phi\rangle = |\mu\rangle\langle\nu|\}$ will be a set of orthogonal unit base vectors. Now we can denote the strategy state of player i by state vector or a density matrix in operator Hilbert space,

$$\rho^i = \sum_{\Phi,\Psi} \rho^i_{\Phi\Psi} |\Phi\rangle\langle\Psi| \quad \text{or} \quad \rho^i = |U\rangle\langle U| = \sum_{\Phi,\Psi} U_{\Phi}U^*_{\Psi} |\Phi\rangle\langle\Psi| \ . \tag{34}$$

In the direct product space of all single-player strategy spaces, state of the whole N-player system is

$$\rho^S = \prod_{i=1}^{N} \rho^i, \tag{35}$$

if the state of all players are independent, or in Game Theory language, non-cooperative. According to Quantum Mechanics, the payoff value should be

$$E^i = Tr\left(\rho^S H^i\right). \tag{36}$$

So a quantum game is defined as

$$\Gamma^q = \left(\prod_{i=1}^{N} \otimes S^q_i, \{H^i\}\right). \tag{37}$$

Some specific quantum games such as Quantum Penny Flip Game[3] and Quantum Prisoner's Dilemma[4] have been reexpressed and studied in the new representation in [1] and [5]. Here we 'theoretically' define a manipulative general quantum game, and prove that it can be described in our new representation. A real quantum game is defined in quantum operator level, so all strategies are quantum operators acting on a quantum object, whose initial state is denoted as $\rho^q_0 \in \mathbb{H}^q$. Here, because we use density matrix, which can also be regarded as an operator, to represent state of the quantum object, $\mathbb{H}^q$ denotes both the state space of the quantum object and operator space on it. Every player choose strategy $\hat{U}^i \in \mathbb{H}^i$, which is an operator from subspace of $\mathbb{H}^q$ onto the subspace, then the jointed operator acting on the quantum object is

$$\hat{U} = \mathcal{L}\left(\hat{U}^1, \cdots, \hat{U}^N\right), \tag{38}$$

a linear mapping of $\hat{U}^i$ from direct product strategy space $\mathbb{H} = \prod_i^N \otimes \mathbb{H}^i$ to operator space of quantum object $\mathbb{H}^q$,

$$\mathcal{L}\left(\cdots, \alpha \cdot \hat{U}^i, \cdots\right) = \alpha \cdot \mathcal{L}\left(\cdots, \hat{U}^i, \cdots\right), \alpha \in \mathbb{C}, \forall i. \tag{39}$$

Product and direct product are typical forms of such mapping[1, 5]. Then the end state of the quantum object is

$$\rho^q = \hat{U}\rho^q_0\hat{U}^{\dagger}. \tag{40}$$

Payoff value of player i is determined by

$$E^i = Tr\left(P^i \rho^q\right), \tag{41}$$

in which P^i is a matrix in $\mathbb{H}^q$, named payoff scale matrix, which gives the rule or scale to determine the payoff. In Quantum Penny Flip Game, it's

$$P^1 = |U\rangle\langle U| - |D\rangle\langle D| = -P^2.$$

For player 1, this means to assign 1 to up state and -1 to down state. The form in Quantum Prisoner's Dilemma Game has also been given in [5]. So a Quantum Game in operator level, the language of Quantum Mechanics, is

$$\Gamma^{q,o} = \left(\mathbb{H}^q, \prod_{i=1}^{N} \otimes \mathbb{H}^i, \mathcal{L}, \left\{P^i\right\}\right). \tag{42}$$

Now as we did in classical game, we need to prove $\Gamma^{q,o}$ can be equivalently described by Γ^q. Before the detailed proof, we want to point out the linear property of $\mathcal{L}$, equ(39), is a very important condition. Even in classical game, we can find the corresponding implied condition. Starting from a pure strategy game, the only thing we know is the elements of payoff tensor G^i, which can only give the payoff for pure strategies, but we need to know the payoff value for mixture strategies. Equ(3) uses mathematical expectation to calculate it. It seems quite natural, but is it really the only possible? The nonlinear behavior might be possible. For example, consider the situation that a girl and a boy work together. They have two optional jobs. When they use a little time together, the interaction between them is weak, the efficiency is low; when they spend more time together, they know each other better, so the efficiency is higher; when they spend too much time together, they will find more shortcoming of each other, or otherwise they will flirt with each other, anyway, the efficiency will be lower again. This is a truly non-linear behavior. Our current Game Theory could never describe this phenomenon. So this is the implied linear condition of classical game,

$$E^i\left(\cdots, \alpha\cdot\rho^i_a + \beta\cdot\rho^i_b, \cdots\right) = \alpha\cdot E^i\left(\cdots, \rho^i_a, \cdots\right) + \beta\cdot E^i\left(\cdots, \rho^i_b, \cdots\right), \forall \alpha, \beta \in [0,1]. \tag{43}$$

This will require $\left\{G^i\right\}$ are linear mappings, $(0, N)$-tensors. Now in our quantum game, this condition implies linear property of mapping $\mathcal{L}$ and equ(41), the trace operator. And in the abstract form Γ^q, this condition is automatically fulfilled when $\left\{H^i\right\}$ are $(1,1)$-tensors. Now we prove the equivalence.

Theorem III $\Gamma^{q,o}$ is equivalent with Γ^q. This is to say for all players with arbitrary strategies, the two representations give the same payoff for every player.

Proof For player i's operator space, choose a set of base vectors according to equ(33), the inner product definition, and denote them as $\left\{|s\rangle^i\right\}$ and $\left\{\hat{s}^i\right\}$, which, for the quantum object, are operators such as $\{|\mu\rangle\langle\nu|\}$, but for the players, are some base strategies. Suppose, player i choose strategy $\hat{U}^i$, which can be expanded as

$$\hat{U}^i = \sum_s U^i_s |s\rangle^i = \sum_s U^i_s \hat{s}^i.$$

Density matrix form of this player's state is

$$\rho^i = \sum_{\phi^i,\psi^i} U^i_\phi \left(U^i_\psi\right)^* \left|\phi\right\rangle^i \left\langle\psi\right|^i .$$

System state density matrix is

$$\rho^S = \prod_i^N \sum_{\phi^i,\psi^i} U^i_\phi \left(U^i_\psi\right)^* \left|\phi\right\rangle^i \left\langle\psi\right|^i .$$

Define every elements payoff matrix in this representation as

$$\left\langle \phi^1 \cdots \phi^N \right| H^i \left| \psi^1 \cdots \psi^N \right\rangle = H^i_{SS'} = Tr\left(P^i \mathcal{L}\left(\hat{\phi}^1, \cdots, \hat{\phi}^N\right) \rho^q_0 \mathcal{L}^\dagger \left(\hat{\psi}^1, \cdots, \hat{\psi}^N\right)\right). \tag{44}$$

Payoff from equ(41) is

$$\begin{aligned}
E^i &= Tr\left(P^i \mathcal{L}\left(\cdots, \sum_{\phi^j} U^j_\phi \hat{\phi}^j, \cdots\right) \rho^q_0 \mathcal{L}^\dagger \left(\cdots, \sum_{\psi^j} U^j_\psi \hat{\psi}^j, \cdots\right)\right) \\
&= Tr\left(P^i \sum_{\phi^j} U^1_\phi \cdots U^N_\phi \mathcal{L}\left(\cdots, \hat{\phi}^j, \cdots\right) \rho^q_0 \sum_{\psi^j} \left(U^1_\psi\right)^* \cdots \left(U^N_\psi\right)^* \mathcal{L}^\dagger \left(\cdots, \hat{\psi}^j, \cdots\right)\right) \\
&= \sum_{\phi^j} U^1_\phi \cdots U^N_\phi \sum_{\psi^j} \left(U^1_\psi\right)^* \cdots \left(U^N_\psi\right)^* Tr\left(P^i \mathcal{L}\left(\cdots, \hat{\phi}^j, \cdots\right) \rho^q_0 \mathcal{L}^\dagger \left(\cdots, \hat{\psi}^j, \cdots\right)\right) . \\
&= \sum_{\phi^j} U^1_\phi \cdots U^N_\phi \sum_{\psi^j} \left(U^1_\psi\right)^* \cdots \left(U^N_\psi\right)^* \left\langle \phi^1 \cdots \phi^N \right| H^i \left| \psi^1 \cdots \psi^N \right\rangle \\
&= Tr\left(\rho^S H^i\right)
\end{aligned}$$

So for all quantum pure strategy, those two forms give the same payoff. We require P^i is hermitian. It's a payoff scale matrix, which assigns values to every state, therefor, in a specific set of base vectors, P^i should have only diagonal terms. So in this representation, $P^i_{\mu\nu} = 0 = \left(P^i_{\nu\mu}\right)^*$, then $P^i = \left(P^i\right)^\dagger$ generally. Therefor, H^i is also hermitian. Further more, in density matrix form, not only pure quantum strategies, but also quantum mixture strategies are allowed to be used by quantum players. And still the payoff are given by equ(36). This means, not only unitary operator, but also mixture of unitary operators, can be a strategy of quantum player. Whether such strategy is applicable or not will depend on further application and realization of the idea of quantum game.

Now classical game and quantum game have the same forms except non-zero off-diagonal terms in quantum payoff matrix. Sometimes, quantum game has classical sub-game, in which a set of special base vectors (strategies) can be found that the corresponding sub-matrix $H^{i,c}$ of H^i is diagonal. An example of this is Quantum Penny Flip Game and Classical Penny Flip Game[1], in which N^c, F^c are classical base vectors while the quantum base vectors include other two base strategies N^q, F^q. In that situation, we say the quantum game has classical limit. So there are several different strategy spaces, classical pure strategy space, classical mixture strategy space, quantum pure strategy space and quantum mixture strategy space. Even a larger space can be took into our consideration, which destroys equ(35), the direct product relation between system state and single-player state. We name it entangled strategy space, which includes vectors in system space not only the direct-product states. And we name the game with diagonal payoff matrix in classical mixture strategy space as Classical Game (CG); the game with diagonal payoff matrix in quantum pure strategy space as Quantized Classical Game (QCG), which are equivalent with CG; the game with general payoff matrix in quantum pure strategy space as Traditional Quantum Game (TQG); the game with general payoff matrix in quantum mixture strategy space as Quantum Game (QG); the game with general payoff matrix in entangled quantum strategy space as Entangled Quantum Game (EQG), the game with diagonal payoff matrix in entangled classical strategy space as Entangled Classical Game (ECG). So

$$CG \subseteq QCG \left(\subseteq ECG\right) \subseteq TQG \subseteq QG \left(\subseteq EQG\right),$$

In which ECG is included in EQG, but not in TQG and QG.

Now our question is what kind of equilibrium states exist in which strategy space. We know Nash Theorem is valid in classical mixture strategy space. Will a Nash-like equilibrium state exist in quantum game, and in which strategy space? A General Classical Nash Equilibrium (GCNE) and General Quantum Nash Equilibrium (GQNE) is defined as

$$E^i\left(\rho^S_{eq}\right) \geq E^i\left(Tr^i\left(\rho^S_{eq}\right)\cdot\rho^i\right), \forall\rho^i, \forall i. \tag{45}$$

When $\rho^S = \prod_i^N \rho^i$, this definition will become equ(4), the traditional definition of NE, but for quantum game, we name it Quantum Nash Equilibrium (QNE).

Proposition NE exists in CG, GCNE exists in ECG, QNE exists in QG and GQNE exists in EQG.

Proof A general proof is still open. The first part is Nash Theorem. For the third part, for a special class of quantum game, in which $\left[H^i, H^j\right] = 0, \forall i, j$, we can prove the existence of QNE. First, find the common eigenvectors set of $\left\{H^i\right\}$, and then in this new representation, all payoff matrix are diagonal. So the quantum game looks similar with classical game, but in another set of base vectors. Because NE exists in the new classical game, QNE exists in the original quantum game. So for such games, QNE exists. For the second and the fourth part, for another special class of quantum game, in which all $\left\{H^i\right\}$ have a common eigenvector ρ^S_M with maximum eigenvalue, so that

$$E^i\left(\rho^S_M\right) \geq E^i\left(\rho^S\right), \forall\rho^S, \forall i, \tag{46}$$

ρ^S_M will be GCNE or GQNE. Since it's a vector in system space, it's possible that it is not a direct product of single player state. Let's keep fingers crossed for a general proof in the near future.

7 Conclusion and discussion

We have seen that our Quantum Game and Entangled Quantum Game are truly independent thing, which is impossible to be put into the old framework of G^i in Γ^c. In fact, elements of G^i are only the diagonal part of H^i. This is partially an answer to the second question in [2], but still not a confirmative answer. The applicability value of this Quantum Game can only been shown through a real quantum game, in which players make use of quantum operators acting on a quantum object. However, most examples we have now are toy games, not from real experiments in quantum world. In the future, we will try to propose one real game from the world of quantum computation or quantum communication, etc.

Now, we turn to discuss the first question in [2], if our new approach is helpful to solve the Classical Game. Since G^i is enough for classical game, theoretically, this new approach brings nothing new into Classical Game. So how about the technical level? Will the new approach be helpful to calculation of NE in Classical Game? We have shown in section §3, a wave vector and its density matrix $\rho^i = \sum_{\mu\nu} \left(\phi^i_\nu\right)^* \phi^i_\mu \left|\mu\right\rangle \left\langle\nu\right|$ can equivalently replace $\vec{p}^i = (\cdots, p^i_\mu, \cdots)^T$. In fact, the former provides much redundant information through the off-diagonal terms, as we know, in classical game, only the diagonal terms $p^i_\mu = \left(\phi^i_\mu\right)^* \phi^i_\mu$ effect the payoff. From theoretical viewpoint, it's waste of time, however, from calculation viewpoint, this means we can use quantum state to calculate problems in Classical Game. Because a single quantum state includes as much as variables in a very complex combination classical state, this makes it possible to improve calculation of NE in Classical Game. Further more, if an evolution equation of wave vector, instead of our pseudo-dynamical equation of density matrix, can be found, next time, when we need to calculate classical NE, the only thing need to do is to choose an arbitrary initial state of a quantum system and let it evolute according to the equation, and then, the end state will be the answer. As in Quantum Mechanics, Schrödinger Equation of wave vector is equivalent with Liouville Equation of density matrix, all linear equation of density matrix can be transferred into linear equation of wave vector. Although our current pseudo-dynamical equation are not linear, it is not impossible to find another better linear equation to describe the evolution process.

Besides the technical level of NE calculation, the new approach opens an exited way to deal with Evolutionary Game. From the application experience in [1], not only the end state, but also the pseudo-dynamical process seems meaningful in Game Theory. And in the new representation, the system state of non-cooperative players is a direct product density matrix, while a general density matrix in system state space includes some correlation between players, can be naturally used to discuss cooperative game. Evolutionary Game and Cooperative Game are another two important topics. In traditional Game Theory, the framework of evolutionary game is far from elegant. It will be a great progress if this new representation can be easily applied onto those two aspects.

For example, let's suppose a GCNE/GQNE, or specially the ρ^M in equ(46) has been found for a game. It's not a direct-product state, so that

$$\rho^S \neq \prod_i^N Tr_{-i}\left(\rho^S\right). \tag{47}$$

But it's still probable in some sectors of players $N = \bigcup_{j=1}^{s} N^j$, in which N is the set of players and N^j is subset of N, that

$$\rho^S = \prod_j^s \rho^j, \tag{48}$$

although when $\left|N^j\right| > 1$, $\rho^j \neq \prod_i^{\left|N^j\right|} Tr_{-i}\left(\rho^j\right)$. So the subsets $\left\{N^j\right\}$ can be regarded as player groups. Therefor, if such groups can be naturally derived from ECG and EQG, our new representation will be a way being applicable to cooperative game.

For the applicability value of our quantum game, another factor has to be took into consideration. Not all vectors in quantum pure strategy space are unitary operators. If we require all physical operators are unitary, only part of the space can give real applicable strategies, and even worse, they might not be a subspace. This will exclude some equilibrium solutions, even when QNE or GQNE does exist in the whole space of quantum strategy space or entangled quantum strategy space. In that situation, it's not a confirmative result even we prove the existence of NE. Fortunately, in a 2×2 quantum operators space, the unitary operators still expand a subspace. A general 2×2 quantum operator is

$$\hat{U} = \xi \cdot \hat{I} + x \cdot \hat{\sigma}_x + y \cdot \hat{\sigma}_y + z \cdot \hat{\sigma}_z, \xi, x, y, z \in \mathbb{C}, \tag{49}$$

while a unitary 2×2 quantum operator is

$$\hat{U} = U = \xi \cdot \hat{I} + ix \cdot \hat{\sigma}_x + iy \cdot \hat{\sigma}_y + iz \cdot \hat{\sigma}_z, \xi, x, y, z \in \mathbb{R}. \tag{50}$$

So we can still ask the question that if QNE and GQNE exist in the subspace. But we only have such good conclusion for 2×2 quantum operator space, we don't know it's a general result or not. However, even if it's not a general conclusion so that we have to discuss equilibrium state in the whole space including non-unitary operators, maybe someday, we can make use of non-unitary operators as strategy so it's even better to consider questions in the whole space. All such open problems depend on the application and realization of real quantum games.

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