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Exact Solution of the Social Learning Model

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Outline

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1. The basic model of social learning and the motivation
2. Previous results and solutions
3. Problems in previous solutions of the game
4. An exact solution of our own
5. Comparison between our solution and the previous solutions on the basic and extended models

Warning: This talk is going to be fairly mathematical, but I promise the ideas will be emphasized.

The questions

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- Question: If every individual is learning the truth about the world individually, can the population finds the true state of the world?
- Examples: Buying an iPhone or an android phone, considering to adopt or not one newly invented technology, choosing restaurants from observing number of customers.

Figure : Lineup before Apple store Shanghai, China

The basic model

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- One state of the world $s_w = \pm 1$, randomly chosen with probability q^{ext} that $s_w = 1$, fixed during the game.
- N learners, every learner receives private signals s^j (with probability $p \geq 0.5$ $s^j = s_w$) and acts in a given sequence, actions of all previous learners are available to all later learners, $\vec{a}^{j-1} = \left(a^1, a^2, \cdots, a^{j-1}\right)$.
- Payoffs are not public knowledge, $p^j = 1$ when $a^j = s_w$ and otherwise $p^j = -1$.
- The jth learner makes a decision according to $\lambda^j = P\left(s_w = 1 | \vec{a}^{j-1}, s^j\right)$.

One simple calculation of λ^j

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- Counted the numbers of 1 and -1 in the action history, N_+^{j-1} and N_-^{j-1}, and then choose to act according to $\max\left(N_+^{j-1} + s^j, N_-^{j-1}\right)$.
- Using this overly simplified strategy, researcher have shown that there are information cascades and cascading towards either the true or the wrong state of the world.

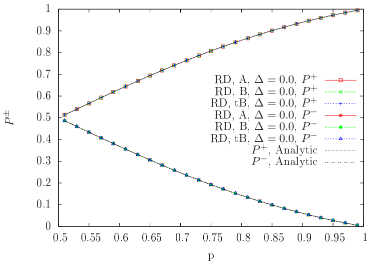

Figure : Probability of right or wrong cascade depends on p, q^{ext}. $s_w = 1$.

To get some intuition

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- Start from the first learner, upon receiving $s^1 = -1$, then $a^1 = -1$.
- Assuming the second learner get $s^2 = 1$, then he is in a tie. Let us assume he breaks it randomly, and result happen to be $a^2 = -1$.
- The third learner receives $s^3 = 1$, but he will choose $a^3 = -1$ according to the overly simplified counting strategy.

Table : Look at only the second row for now

$\vec{s}$	-1	1	1	1
$\vec{a}, \lambda^{j,B}$	-1	-1	-1	-1
$\vec{a}, \lambda^{j,tB}$	-1	-1	-1	-1
$\vec{a}, \lambda^{j,A}$	-1	-1	1	1

Other interesting open questions

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- How large is the difference if I choose to act near the beginning or the end? Should I pay for that?
- If I am trying to popularize my product or my book, how much copies of the book should I secretly purchase?
- May I do better than the counting strategy? It has been shown that believing in oneself is better than the random tie break.
- Is it possible for me to take into consideration that when $\vec{a}^{3-1} = (-1,-1)$, it is possible that $\vec{s}^{3-1} = \{(-1,-1),(-1,1)\}$? **This is the key question which leads to our exact solution.**

$a^j\left(\vec{a}^{j-1}, s^j\right)$ from counting or Bayesian

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- Counting strategy implies that $a^j = \left(N_+^{j-1} + s^j > N_-^{j-1} ? 1 : -1\right)$
- Maybe we can try a Bayesian analysis,

$$\begin{aligned}
\lambda^j = P\left(s_w = 1|\vec{a}^{j-1}, s^j\right) &= \frac{P\left(s_w = 1, s^j|\vec{a}^{j-1}\right)}{P\left(s_j|\vec{a}^{j-1}\right)} \\
&= \frac{P\left(s^j|s_w = 1, \vec{a}^{j-1}\right) P\left(s_w = 1|\vec{a}^{j-1}\right)}{P\left(s_j|\vec{a}^{j-1}\right)} \\
&= \frac{P\left(s_w = 1|\vec{a}^{j-1}\right) P\left(s^j|s_w = 1\right)}{P\left(s^j|s_w = 1\right) P\left(s_w = 1|\vec{a}^{j-1}\right) + P\left(s^j|s_w = -1\right) P\left(s_w = -1|\vec{a}^{j-1}\right)}.
\end{aligned} \tag{1}$$

where everything except $P\left(s_w = 1|\vec{a}^{j-1}\right)$ is known already.

Twisted Bayesian, continued

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- This twisted Bayesian analysis uses

$$\begin{aligned} P\left(s_w = 1|\vec{a}^{j-1}\right) &= P\left(s_w = 1|\vec{a}^{j-2}, s^{j-1}\right) \\ &= P\left(s_w = 1|\vec{a}^{j-2}, s^{j-1} = a^{j-1}\right) \\ &= \lambda^{j-1}\left(\vec{a}^{j-2}, s^{j-1} = a^{j-1}\right) \end{aligned} \quad (2)$$

  so that $\lambda^{j-1} \longrightarrow \lambda^{j}$ is a complete formula.
- But how can we effectively assume that $s^{j-1} = a^{j-1}$ (s^{j-1} **is unknown to learner** j) upon observing a^{j-1} (**known to learner** j)?

Twisted Bayesian, continued

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- There is no justification in assuming that $s^{j-1} = a^{j-1}$ at all.
- In a sense, since we assume that $s^{j-1} = a^{j-1}$ upon observing a^{j-1}, this twisted Bayesian is no better than the counting strategy, where actions are treated like signals.
- If signals are public knowledge, then indeed counting strategy is correct:

$$\vec{s}^{j-1} = \{1, -1, 1, -1, -1, -1, 1, \ldots\} \tag{3}$$

Another Bayesian, the idea

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- The idea is to convert $\vec{a}^{j-1} \Longrightarrow \left(\vec{s}^{j-1}, P\left(\vec{s}^{j-1}\right)\right)$, then calculate $\lambda^j\left(\vec{s}^{j-1}, s^j\right)$ for each $P\left(\vec{s}^{j-1}\right)$ and then combine them to find λ^j.
- Upon observing $\vec{a}^{3-1} = (-1,-1)$,

$$P\left(\vec{s}^{3-1} = (-1,-1) \,|\vec{a}^{3-1} = (-1,-1)\right) = \frac{2}{3}, \tag{4a}$$

$$P\left(\vec{s}^{3-1} = (-1,1) \,|\vec{a}^{3-1} = (-1,-1)\right) = \frac{1}{3}, \tag{4b}$$

$$P\left(\vec{s}^{3-1} = (1,-1) \,|\vec{a}^{3-1} = (-1,-1)\right) = 0, \tag{4c}$$

$$P\left(\vec{s}^{3-1} = (1,1) \,|\vec{a}^{3-1} = (-1,-1)\right) = 0. \tag{4d}$$

- This is potentially different from considering only $\vec{s}^{3-1} = (-1,-1)$, which is the case of the counting strategy and also the twisted Bayesian

Another Bayesian, formal expressions

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- The above idea becomes

$$\lambda^j = \frac{P\left(\vec{a}^{j-1}, s^j | s_w = 1\right) P\left(s_w = 1\right)}{P\left(\vec{a}^{j-1}, s^j | s_w = 1\right) P\left(s_w = 1\right) + P\left(\vec{a}^{j-1}, s^j | s_w = -1\right) P\left(s_w = -1\right)} \tag{5}$$

where $P\left(\vec{a}^{j-1}, s^j | s_w = 1\right) = P\left(\vec{a}^{j-1} | s_w = 1\right) P\left(s^j | s_w = 1\right)$

- So only unknown is $P\left(\vec{a}^{j-1} | s_w = 1\right)$, but a^{j-1} is determined by λ^{j-1}, so in a sense we have complete formula $\lambda^{j-1} \longrightarrow \lambda^j$ without assuming anything.

On the original model

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- Let us now compare the three.

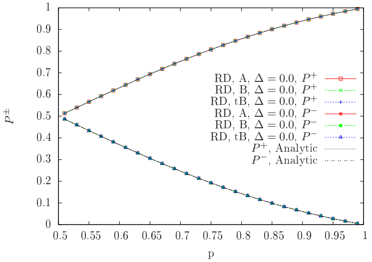

Figure : There is no difference at all

- So our solution is just in principle better than the other two, but not practical difference. Why?

On the extended model

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- Values of $\lambda^{j,A}$ is different from those of $\lambda^{j,tB}$ and $\lambda^{j,A}$, for example, when $q^{ext} = 0.5, p = 0.7$,

$$\lambda^{3,A}\left(s_w = 1|a^{12} = -1 - 1, s^3 = 1\right) = 0.43 \quad (6)$$
$$\lambda^{3,B,tB}\left(s_w = 1|a^{12} = -1 - 1, s^3 = 1\right) = 0.3 \quad (7)$$

- Both less than 0.5, so $a^3 = -1$ will be chosen. No difference in actions
- $1 - 0.43 = 0.57$ is less likely than $1 - 0.3 = 0.7$, can we make use of this information?
- What if we decide not to act when $\left|\lambda^{j,A} - 0.5\right| < \Delta$. So we extended the model by introducing a level of reservation, Δ

On the extended model

- We may now take another look at Table 1, which is reproduced in the following

Table : Actions due to $\lambda^{j,A}$ is different from the other two

$\vec{s}$	-1	1	1	1
$\vec{a}, \lambda^{j,B}$	-1	-1	-1	-1
$\vec{a}, \lambda^{j,tB}$	-1	-1	-1	-1
$\vec{a}, \lambda^{j,A}$	-1	-1	1	1

On the extended model,continued

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- Extended model with $\Delta \neq 0$,

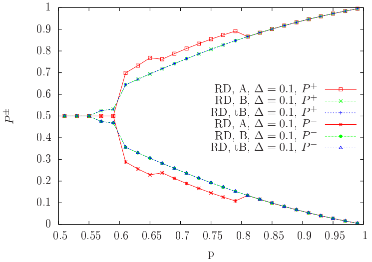

Figure : There is a difference in actions when $\Delta \neq 0$

- Using $\lambda^{j,A}$, Probability of cascading when $\Delta \neq 0$ towards the true state is higher than that of $\Delta = 0$
- Using $\lambda^{j,B,tB}$, even when $\Delta \neq 0$, Probability of cascading is the same as that of $\Delta = 0$

Conclusions and discussion

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- $\lambda^{j,A}$ is different from $\lambda^{j,B}$ and $\lambda^{j,tB}$ both in principle and in practice when $\Delta \neq 0$
- $\lambda^{j,A}$ is better than the other two in the sense that: Firstly, it is generally applicable to the social learning game even with other extra modifications; Secondly, learners using it overall achieve larger payoffs.
- Other investigations making use of the counting strategy or the twisted Bayesian analysis can be revised to be based on this new strategy evolution process

Time for questions

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- Question time